

## *A note on incompleteness and heterologicality*

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It is a well known aspect of Frege's conception that the incompleteness of predicative or functional symbols precisely parallels the incompleteness of the entities they symbolize. This thought can be amplified in various ways. One might hold that a predicate, being a symbol for a concept, a function mapping objects to truth values, is itself a species of function, a linguistic function mapping names to propositions. This way of amplifying the thought is due to Geach (1973); its coherence is defended in Sullivan 1991. Alternatively, one might hold that a property of objects, or a relation between objects, is symbolized by a property of the name of that object, or

by a relation between the names of those objects. According to this way of amplifying the thought, conceived by Wittgenstein, what symbolizes that the object  $a$  stands in a certain relation to object  $b$  is the fact that the name ‘ $a$ ’ stands in a certain conventionally associated relation to the name ‘ $b$ ’: ‘thus facts are symbolized by facts, or more correctly: that a certain thing is the case in the symbol says that a certain thing is the case in the world’ (NB 96; cf. TLP 3.14, 3.1432). Or somewhat differently again, one might hold that a predicate is a feature or property of a proposition (Geach 1972; Dummett 1981). For some purposes the differences between these various ways of expanding on Frege’s basic thought are important, but all that matters here is what I shall continue to call the basic thought itself: that incompleteness in symbols parallels the incompleteness in what they symbolize, so that symbols manifest distinctions of type and level parallel to those among what they symbolize (or, as I shall also say, what they refer to). It might also be important in other contexts to ask whether the basic thought involves an asymmetry: whether, that is, incompleteness is primarily a feature of the entities symbolized and merely reflected in their symbols, or whether it must be understood in the first instance as a feature of symbols and projected onto what they symbolize. But I have chosen to talk neutrally of a ‘parallel’ incompleteness to indicate that, for present purposes, this question can be left undecided.

It is also well known that the idea of a parallelism of type between symbols and their referents played an important role in Frege’s response to the paradoxes. In his letter to Frege in 1902 Russell presented two versions of his paradox. The first and fatal version concerned a class  $r$  defined to have as members classes not members of themselves:

$$\begin{array}{ll} \textit{Class version} & x \in r \text{ iff } \sim x \in x \\ \text{so that, in particular,} & r \in r \text{ iff } \sim r \in r. \end{array}$$

The second concerned a property  $R$  defined to apply to just those properties that do not apply to themselves:

$$\begin{array}{ll} \textit{Property version} & R(F) \text{ iff } \sim F(F) \\ \text{so that, in particular,} & R(R) \text{ iff } \sim R(R). \end{array}$$

Though it did little to soften the blow of the first, Frege was able immediately to dismiss the second version of the paradox: ‘a predicate is as a rule a first-level function which requires an object as argument and which cannot therefore take itself as argument (subject)’ (Frege 1980: 132). In the purported definition of ‘ $R$ ’ the string ‘ $\sim F(F)$ ’ is, simply, meaningless. If the outer ‘ $F( )$ ’ has an incompleteness suitable to accommodate its argument, then, if it is to be the same symbol, so must the inner ‘ $F$ ’ – giving the unfinished ‘ $\sim F(F( ))$ ’. Alternatively, if the inner ‘ $F$ ’ has no such incompleteness, then, if it is supposed to be the same symbol, neither has the outer ‘ $F$ ’ – yielding the list-like ‘ $\sim FF$ ’. The basic thought that symbols share the



tent use for ‘X’, and then have been led to put the blame on ‘ $\varphi$ ’. Similarly, nothing turns on the value assumed for  $n$  – whether we imagined  $\varphi$  to be of first level, or second level, or whatever. Independent of such details, the point is simply that we cannot construe both ‘X’ and ‘ $\varphi$ ’ consistently: the first conjunct demands that they be of the same level, while the second demands that they not be.<sup>2</sup> So the definition fails of any reading. There is no such condition as being heterological.

‘Yet surely’ – one imagines someone reacting – ‘it is straightforwardly true that, while “short” is short and “familiar” familiar, “long” is not long and “unfamiliar” is not unfamiliar; the respect in which the second pair contrast with the first is obvious; and to be heterological is just to be like “long” and “unfamiliar” in that respect.’ Which respect? If the feature that ‘long’ and ‘unfamiliar’ supposedly share cannot be intelligibly specified, then it is no more than illusion that there is such a feature. And, given Frege’s basic thought about type-parallelism, that is just what the above arguments show.

This blank response need not deny the objector’s starting points. It can perfectly well be allowed that ‘short’ is short while ‘long’ is not long. That is to say, in Wittgenstein’s terminology, that the *sign* ‘short’ is short, and the *sign* ‘long’ is not long. It is not to say that the first refers to a property it has while the second refers to a property it lacks. A sign, being an object, may well be short (or long). But being an object, a sign is not what refers to this property. More generally, a mere sign is *never* that to which a reference is ascribed (cf. TLP 3.326–3.327).

The concessions just made readily suggest, however, an attempted reformulation of the paradox. Suppose we grant that the sign ‘short’ (or ‘long’) is not, all by itself, what refers to the property of being short (or long). Still, it must be admitted that there is *some* connection between the sign and the property. A sign, Wittgenstein said, is ‘the part of the symbol perceptible by the senses’ (TLP 3.32). The sign, we might say, is a sensible indicator, or *index* (cf. Long 1969) of the occurrence of the symbol. So, even accepting everything so far said, won’t the paradox return if we exploit that connection to redefine ‘heterological’ as applying to signs?

<sup>2</sup> To reiterate, neither conjunct is claimed to be problematic on its own. Some have held, in connection with the so-called ‘concept *horse*’ problem, that Frege’s notion of incompleteness already puts us in a bind in relation to the first conjunct: how, it is asked, are we to complete the schema ‘X refers to  $\varphi$ ’ without replacing ‘X’ and ‘ $\varphi$ ’ by expressions which, as grammatically singular terms, frustrate our intentions? Answer: by getting rid of ‘refers to’, which was in the first place no more than a stand in. What it stands in for depends on the intended type of ‘X’ and ‘ $\varphi$ ’. If they are of first level, for instance, one might say: any sentence  $Xn$  is true if what  $n$  refers to is  $\Omega$  (where ‘ $n$ ’ ranges over actual and possible names, and both ‘X’ and ‘ $\varphi$ ’ appear with their argument places).

A sign is heterological iff it is the index of a symbol for a property it lacks.

The answer is ‘No’. In sketching above the derivation of the paradox from the usual (purported) definition we appealed to the assumption that ‘Het’ was unambiguous. The attempt to derive a contradiction from this new definition would be blocked by the failure of a corresponding condition: the sign-symbol connection it exploits is *not* one-one. ‘Two different symbols can have the same sign (the written sign or the spoken sign) in common’ (TLP 3.321). A sign’s being the index of a symbol for a property it lacks does not exclude its also being the index of a symbol for a property it has. So, with the new definition, we cannot be pressed to the point of paradox by the question: Is ‘heterological’ heterological or autological? It could perfectly well be both.

It is worth pursuing the argument through another twist that this response might suggest.<sup>3</sup> The sign-symbol relation is not by nature one-one. But confusions that can be engendered by the same sign’s taking on different symbolic roles give a practical reason for contriving a limited or artificial language in which that condition is met, ‘a symbolism which excludes [such confusions] by not applying the same sign in different symbols’ (TLP 3.325). In deference to the idea of a ‘logically perfect language’ that it suggests, call the signs of such a contrived language ‘perfect signs’. Could we not relaunch the paradox by confining the previous definition to such signs, or by introducing a heterologicality predicate into such a language?

A perfect sign is perfectly-heterological iff it is the index of a symbol for a property it lacks.

The answer, again, is ‘No’. ‘Perfectly-heterological’, as defined, draws a genuine distinction amongst pure signs which, let us suppose,<sup>4</sup> puts ‘monosyllabic’ on one side (+ve) and ‘polysyllabic’ on the other (–ve). But again we cannot be pressed into paradox by the question: To which side of the division is ‘perfectly-heterological’ assigned? We’re perfectly free to answer: Neither. That ‘perfectly-heterological’ is defined *over* pure signs does not guarantee that it is one. Equivalently, that the language to which this definition is added was so-far perfect does not guarantee that the expanded language is so. The ‘paradox’ can then assume the innocent form of a proof that it is not. (So, you might conclude, it shows there are some

<sup>3</sup> I am grateful to Roger White for a criticism that persuaded me of this.

<sup>4</sup> We are supposing, that is, that ‘monosyllabic’ and ‘polysyllabic’ are perfect signs; I have switched to these two because the corresponding assumption about the previous examples, ‘long’ and ‘short’, would be too patently false.

things you'd better not try to say *if* you want your language to be perfect. Why, or whether, you should want that is another matter.)

Unsurprisingly, Frege himself never mentioned the heterologicality paradox. Grelling formulated it in 1908; Frege's fullest consideration of the paradoxes dates from 1902–4. I suspect that, if he came across it later, Frege would have dismissed it immediately as too patently trivial a twist on Russell's paradox to command attention. But had he paused to think of it at all, then I'm sure that the response outlined here would very soon have occurred to him. For it can even *seem* that this predicate version of the paradox is genuinely distinct from, and so not already resolved along with, the property version *only if* one assumes that predicates are objects. From the opposed assumption, immediate in Frege's basic thought about incompleteness, there really is no distinct heterologicality paradox at all.

I share the view I said would have been Frege's, that the heterologicality paradox is too trivial a twist to worry about. So the point of this note is not to recommend a solution to it. (That would in any case have called for a *defence* of Frege's basic thought, and obviously I've attempted no such thing.) My intention has been instead to highlight one small reason for questioning the division of the paradoxes made standard by Ramsey: on the one hand, the genuinely logico-mathematical paradoxes, whose resolution involves a simple type structure; and on the other the semantic paradoxes, demanding some species of ramification.<sup>5</sup> On a Fregean approach, heterologicality will not classify in this scheme: it is, surely, a semantic matter, yet the simple hierarchy resolves it. Whatever the value of Ramsey's approach in itself, I believe that the case of heterologicality illustrates that his scheme can as often distort as assist our understanding of his predecessors' attempts to come to grips with the lesson of the paradoxes.<sup>6</sup>

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<sup>5</sup> In Sullivan 2000 I discuss Wittgenstein's treatment of a far more important problem case for Ramsey's classification, namely the propositional version of the paradox posed by Russell in Appendix B of *The Principles of Mathematics*.

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